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Sampling rare fluctuations of height in the Oslo ricepile model

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Abstract

We describe a new Monte Carlo algorithm for studying large deviations of the height of the pile from its mean value in the Oslo ricepile model. We have used it to generate events which have probabilities of order 10^{-1000}. These simulations check our qualitative argument (Pradhan P and Dhar D 2006 *Phys. Rev.* E **73** 021303) that in the steady state of the Oslo ricepile model the probability of large negative height fluctuations $\Delta h = -\alpha L$ about the mean varies as $\exp(-\kappa\alpha^4 L^3)$ as $L \to \infty$ with α held fixed and $\kappa > 0$.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Large deviations of fluctuations in a system have been studied extensively [1, 2] and recently have attracted much attention especially in the non-equilibrium stationary states of driven systems [3–6]. In a recent paper, we have argued that in the critical slope-type stochastic toppling models in d dimensions the probability of large negative height fluctuations $\Delta h = -\alpha L$ about the mean for a system of size L decays superexponentially, as $\exp\left(-\kappa\alpha^{\frac{d+2}{1-\omega}} L^{d+2}\right)$, for $L \to \infty$, with $\alpha > 0$ held fixed [7]. Here, $\kappa > 0$ and ω is an exponent defined by $\langle(\Delta h)^2\rangle \sim L^{2\omega}$. Since the arguments are plausible but not rigorous, it seems worthwhile to check them by numerical simulations. However, straightforward sampling techniques fail in this case, as the probabilities become very small even for fairly small L. For example, for $L = 10$, in $d = 1$, already the probability of the minimum slope configuration is of order 10^{-45}.

In this paper, we numerically estimate the probabilities of large negative height fluctuations of the pile in the steady state in the one-dimensional Oslo ricepile model using a variation of the general 'go with the winners' strategy [8], adapted to our problem. We estimate the full probability distribution function $\mathrm{Prob}_L(\Delta h)$ where Δh is fluctuation of the height of the pile about its mean value. This distribution has a scaling form $L^{-\omega} g(\Delta h/L^{\omega})$. The 1D Oslo

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model is expected to be in the Edwards–Wilkinson universality class, for which $\omega = 1/4$ [9]. Our non-rigorous arguments in [7] imply that in this case the scaling function should vary as $\exp(-\kappa x^4)$ for $x \ll -1$ where $\kappa > 0$. Our numerical data fully supports the theoretical expectation.

2. Definition of the model

The Oslo ricepile model [10, 11] is a stochastic sandpile-like model defined on a one-dimensional lattice with a critical threshold value for the slope above which a toppling occurs, and the threshold is randomly reset after each toppling. Here, we use an equivalent version of the rules as given in [12]: we consider a chain of length L. A configuration of the pile is specified by an integer height variable h_i at each site i. The slope z_i at site i is defined to be $h_i - h_{i+1}$, with $h_{L+1} \equiv 0$. Any site i with slope $z_i \leqslant 1$ is stable. Any site i with slope $z_i \geqslant 3$ is said to be unstable and relaxes by toppling. Slope 2 can be either stable or unstable. Whenever the slope at a site reaches the value 2 from a different value, because of incoming or outgoing grains, it is created as an unstable 2 (denoted by $\bar{2}$). A $\bar{2}$ becomes stable 2 (denoted by 2 without overbar) with probability p without any toppling, or it topples with probability $q = 1 - p$. Whenever there is a toppling at site i, one grain is moved from the site i to $i + 1$. If there is a toppling at the right end $i = L$, one grain goes out of the system. Grains are added only at site 1 and only after avalanche caused by the previous grain has stopped.

The 1D Oslo ricepile model has a remarkable Abelian property that the final height configuration does not depend on the order we topple the unstable sites [12]. Also, after an addition of $L(L + 1)$ grains to any configuration, the probabilities of different stable configurations are exactly the same as in the steady state, independent of the initial configuration [12]. The number of recurrent stable configurations in the steady state can be calculated exactly and is approximately $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{2L+1}$ for large L [13]. In the steady state height profile fluctuates with an average slope between 1 and 2. The height h_1 at the site 1 has a stationary probability distribution, $\mathrm{Prob}_L(h_1)$, which is sharply peaked near its average value $\bar{h}_1$. For large system size L, the average height $\bar{h}_1$ varies linearly with L, and the fluctuations in h_1 scale as a sublinear power of L, with variance of h_1 varying as $L^{2\omega}$, with $0 < \omega < 1$.

3. Exact calculation of $\mathrm{Prob}_L(h_1)$ for small L

The probability distribution of h_1 in the steady state can be exactly calculated numerically for small L using the operator algebra satisfied by addition operators [12]. We recapitulate this briefly here. We denote any configuration by specifying slope values at all sites from $i = 1$ to $i = L$ by a string of L integers (with or without overbar), e.g., $|10\ldots\bar{2}2\rangle$. For unstable site $1 < i < L$, the rules are as given below:

$$|\ldots, z_{i-1}, \bar{2}, z_{i+1}, \ldots\rangle \to p|\ldots, z_{i-1}, 2, z_{i+1}, \ldots\rangle + q|\ldots, \overline{(z_{i-1}+1)}, 0, \overline{(z_{i+1}+1)}, \ldots\rangle$$
$$|\ldots, z_{i-1}, \bar{3}, z_{i+1}, \ldots\rangle \to |\ldots, \overline{(z_{i-1}+1)}, 1, \overline{(z_{i+1}+1)}, \ldots\rangle. \quad (1)$$

We use the convention $\bar{1} = 1$ and $\bar{3} = 3$. As a slope 1 is always stable, and slope 3 is always unstable, the overbar may be omitted without any confusion. At the left end, rules are as given above except that there is no left neighbour of site 1. At the right end $i = L$, the rules are

$$|\ldots, z_{L-1}, \bar{2}\rangle \to p|\ldots, z_{L-1}, 2\rangle + q|\ldots, \overline{(z_{L-1}+1)}, 1\rangle$$
$$|\ldots, z_{L-1}, 3\rangle \to |\ldots, \overline{z_{L-1}+1}, \bar{2}\rangle. \quad (2)$$

Using these toppling rules repeatedly and the Abelian property, we can relax any unstable configuration.

Table 1. Table for the probability of the least mass configuration for $L = 2, 3, 4$.

L	Probability of the least mass configuration
2	$(1+p)q^4$
3	$(1+4p+6p^2+5p^3+2p^4)q^{10}$
4	$(1+10p+45p^2+125p^3+241p^4+341p^5+369p^6+307p^7+190p^8+81p^9+18p^{10})q^{20}$

Let us now consider the state vector $|\bar{2}\bar{2}\ldots\bar{2}\bar{2}\rangle$ where all the slopes are unstable 2's. If we add one more grain at site $i = 1$ in this state, we get the same vector back by toppling only at the sites with $z_i = 3$ repeatedly. This implies that the vector obtained by relaxing this vector fully is an eigenstate of the evolution operator with eigenvalue 1, and as the steady state is unique this is the unique steady-state vector. If we relax this unstable configuration fully, we get all the different recurrent stable configurations, each with probability same as in the steady state. For example, if we relax $|\bar{2}\bar{2}\rangle$ for $L = 2$, we get the following sequence:

$$|\bar{2}\bar{2}\rangle \to p|2\bar{2}\rangle + q|1\bar{2}\rangle \to p^2|22\rangle + pq|12\rangle + pq|1\bar{2}\rangle + q^2|\bar{2}1\rangle \to \cdots \to$$
$$p^2|22\rangle + (p+p^2)q|12\rangle + (p+p^2)q^2|21\rangle + (p+p^2)q^3|02\rangle + (1+p)q^4|11\rangle.$$

One can similarly calculate the steady-state probabilities for higher L. In table 1, we list the resulting expression for the probability of the configuration $|11\ldots11\rangle$ with all $z_i = 1$ for $L = 2, 3, 4$. For larger L, the calculation becomes very tedious. There are two branches for relaxing any unstable site with $z = \bar{2}$ and the total CPU time increases as $\exp(L^3)$ as $\mathcal{O}(L^3)$ relaxations of unstable sites are required to reach the steady state[1]. The steady-state probabilities for L up to 8 were calculated earlier by Corral [15]. We have calculated the probability of the configuration $|11\ldots11\rangle$ exactly numerically for $L \leqslant 12$ using a simple code written in C. This configuration has the least slope, and the least mass amongst all the recurrent configurations. We shall call it the least mass configuration. We used specific numerical values $p = q = 1/2$, to simplify the calculation, so that the probabilities are simple numbers and not polynomials in p. Even then, for $L > 12$, the computer time required becomes prohibitive.

For large negative deviations, the probability becomes very small. Even for system size as small as $L = 6$ and $L = 7$, the probabilities of the minimum mass configuration are 4.81×10^{-11} and 1.76×10^{-15}, respectively. For $L = 12$, this probability is 1.23×10^{-55}. Clearly, this probability tends to zero as q tends to zero. We argued in [7] that this probability is $\mathcal{O}(q^m)$ with $m = L(L+1)(L+2)/6$, for $q \to 0$, and is expected to decrease as $\exp(-\kappa(q)L^3)$ for all q, with $\kappa(q)$ approximately equal to $\frac{1}{6}\log(1/q)$ for small q. In figure 1, we have plotted negative of logarithm of the probability of the least mass configuration versus $L(L+1)(L+2)$ for $q = 1/2$. The linear increase is in agreement with the theoretical expectation.

4. An unbiassed algorithm for the Oslo model

We start by describing a simple Monte Carlo algorithm to estimate probabilities of low-slope configurations in the steady state using unbiassed sampling. We start with a configuration with all sites unstable and all $z_i = \bar{2}$. There is a random number, uniformly distributed between 0 and 1, at each site i. After each toppling, the random number at the site is replaced by a new random number uniformly distributed between 0 and 1. Let the value of this random number at site i at the end of the update step t be $x(i, t)$. At $t = 0$, all $x(i, 0)$ are independent random variables lying between 0 and 1.

[1] This time can be reduced to $\exp(L^2)$ if we store the weights of $\exp(L)$ configurations to collect terms together.

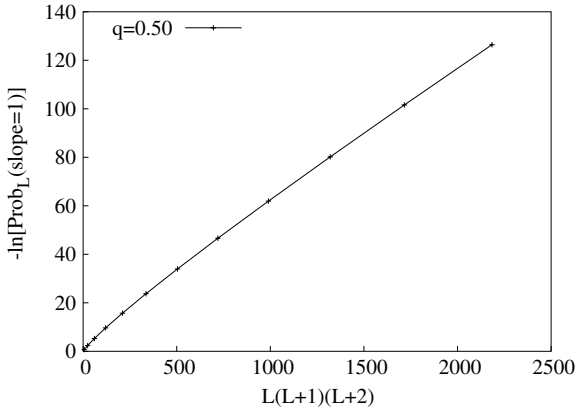

Figure 1. Logarithm (base 10) of the probability of occurrence of the minimum slope configuration, calculated exactly numerically, is plotted against system size $L(L+1)(L+2)$ in the Oslo ricepile model.

As it does not matter in which order we topple the unstable sites, we choose the following rule: at the beginning of any update step (all the unstable sites are with slope $\bar{2}$), *we topple at the site having the largest random number*, if the random number is greater than p. If the number is less than p, the avalanche stops. If this toppling produces any sites with slope 3, we topple at these sites and reset the random number at that site. This is continued till there are no sites left with slope 3. This constitutes the end of one update step. Note that one update step may involve topplings at more than one sites, but only one of these had slope $\bar{2}$ just before toppling, all others had slope 3.

It is clear that this algorithm produces different configurations with correct probability weights.

The evolution of an avalanche up to time t can be specified by a sequence $\mathcal{T}_t$ giving for each update step up to time t, the site with slope $\bar{2}$ selected for toppling and the random number at that site at the time of toppling. Let $j(t)$ be the $\bar{2}$ site selected for toppling at the tth update step and $y(t)$ be the value of the random number at $j(t)$ at the time. Then $\mathcal{T}_t = \{[j(r), y(r)] : r = 1 \text{ to } t\}$. The sequence $\mathcal{T}_t$ will be called the history of the avalanche up to time t.

Our rule of selecting the unstable $\bar{2}$ site for toppling introduces correlations between different $x(i,t)$: if we know the random number at the site selected for toppling, random number at all other non-selected unstable sites must be smaller. It turns out to be more convenient to change to different set of variables $\{x_{\max}(i,t)\}$. Here, $x_{\max}(i,t)$ is defined as the maximum possible value of $x(i,t)$, consistent with a given history $\mathcal{T}_t$. At the beginning, $x_{\max}(i, t=0) = 1$ for all i. The rule for updating this array as the avalanche proceeds is simple: after each toppling, the random variable at the site at the site of toppling is replaced by a new random number, and hence $x_{\max}$ at that site is reset to 1. If there is a toppling at a site with slope $\bar{2}$, and the value of the random number at the toppling site is found to be ξ (this is generated by a stochastic process in our algorithm), then at all other unstable sites j we reset $x_{\max}(j)$ to ξ if its current value is greater than ξ. If the current value is less than ξ, it is left unchanged.

During an avalanche, if we keep track of the variables $x_{\max}(i,t)$, we need not to maintain and update the array $x(i,t)$. Knowing $\mathcal{T}_t$ up to time t fully determines the set of unstable sites

$\mathbf{U}(t)$, out of which the site with the maximum random number has to be selected at update step $t+1$ and also the values of $x_{\max}(i,t)$ for all sites i. This in turn determines the probability distribution of $y(t+1)$ and the probability distribution of the position of toppling $j(t+1)$.

It is straightforward to determine the conditional joint probability distribution of $[j(t+1), y(t+1)]$, using the information in $\mathcal{T}_t$. The conditional probability distribution that $x(j,t)$ is $\leqslant x$, given the value of $x_{\max}(j,t)$, is

$$\text{Prob}(x(j,t) \leqslant x | x_{\max}(j,t)) = g\left(\frac{x}{x_{\max}(j,t)}\right) \tag{3}$$

where $g(\xi) = \xi$, for $0 \leqslant \xi \leqslant 1$, and $g(\xi) = 1$ for $\xi > 1$. As there is no correlation between the values $x(j,t)$ for different j's for the same time t, beyond that implied by the conditions that $x(j,t) \leqslant x_{\max}(j,t)$, we must have

$$\text{Prob}(y(t+1) \leqslant y | \mathcal{T}_t) = \Phi_t(y) = \prod_{j \in \mathbf{U}(t)} g\left(\frac{y}{x_{\max}(j,t)}\right). \tag{4}$$

It is straightforward to generate a random number with a given distribution of the form equation (4) and to select the site for toppling amongst the unstable $\bar{2}$ sites, with the correct probability weights. The detailed algorithm is given in the appendix. Using this algorithm, we can generate $\{j(t+1), y(t+1)\}$, given $\mathcal{T}_t$, and hence recursively generate the complete avalanche.

We illustrate this procedure by a simple example. Consider a ricepile with $L = 6$. At $t = 0$, we have $\mathbf{U}(0) = \{1, 2, 3, 4, 5, 6\}$, as all sites are unstable, and $x_{\max} = 1$ for all sites. In this case, the probability distribution of $y(1)$ is given by

$$\text{Prob}(y(1) \leqslant y) = y^6, \qquad \text{for} \quad 0 \leqslant y \leqslant 1. \tag{5}$$

This can be generated as follows: select a random number z uniformly distributed between 0 and 1, and put $y(1) = z^{1/6}$. If $y(1) < p$, the avalanche stops. If not, we choose $j(1)$ as one of the sites from $\mathbf{U}(0)$ at random, with equal probability. Say, we get $j(1) = 2$. Then, we assign $x_{\max}(j,1) = y(1)$ for all $j \in \mathbf{U}(0)$, except $j = 2$, for which we set $x_{\max}(2,1) = 1$ and topple at site 2. We then topple at any sites with slope 3, and reset the $x_{\max}$ at that site to 1. Finally, we get the configuration with $\mathbf{U}(1) = \{2, 3, 4, 5, 6\}$, and $x_{\max}$ at all these sites is 1. Then, from equation (4),

$$\text{Prob}(y(2) \leqslant y) = y^5, \qquad \text{for} \quad 0 \leqslant y \leqslant 1. \tag{6}$$

We generate a random value with this distribution and set $y(2)$ equal to that value. If $y(2) < p$, the avalanche stops. If not, we choose $j(2)$ at random from $\mathbf{U}(1)$, say $j(2) = 4$. Toppling at this site, and then toppling at sites with slope 3, finally we get the configuration of unstable sites $\mathbf{U}(2) = \{1, 3, 4, 5, 6\}$, and $x_{\max}$ is reset to 1 at all these sites. We now generate the variable $y(3)$, which turns out to have the same distribution as $y(2)$. Now, if $y(3) > p$, we choose a site from $\mathbf{U}(2)$ and so on.

5. Modified algorithm for biassed sampling

If the unbiassed algorithm outlined above is repeated many times, the fractional number of configurations with a given value of h_1 gives an estimate of the corresponding probability. However, this method is clearly unsuitable for estimating probabilities which are much smaller than 10^{-10}. For estimating quantities like the probability of the minimal slope configuration in the steady state, this method is useless for $L > 6$ or so.

Clearly, we need to implement some sort of biassed sampling. In the simplest implementation, one thinks of different possible configurations in the course of evolution

of an avalanche at each step t as a branching tree. If we reach a configuration $\mathcal{C}_t$ at a node at the tth step, the probability of the process stopping is, say, $a(\mathcal{C}_t)$. If we want to sample the low-slope configurations, we do not want the process to die too soon. Then we do not select any of the nodes that correspond to stable configurations and select one of the remaining branches with probability equal to their original probability, divided by the factor $[1 - a(\mathcal{C}_t)]$, and the final survival probability is estimated by product of such factors. This is the basic idea of 'enrichment' in the 'go with the winners' strategy [8].

However, this procedure also is not satisfactory for our problem as there are some unstable nodes all of whose possible resulting stable configurations have heights greater than the minimum height. For example let us consider a case for $L = 5$ where $|21102\bar{2}\rangle$ has two descendants $|21102\rangle$ and $|21111\rangle$, both stable. In the branching tree, these are like leaf nodes. The relaxation process will die after one step if we encounter any such unstable configurations. But it is difficult to identify these directly and avoid them, without a computationally expensive depth search. So the resulting process still has a nonzero probability of reaching such a node at the next step, and the overall probability of survival still decreases exponentially with the depth of the tree.

However, in our algorithm, working with $\{x_{\max}(i, t)\}$, it is straightforward to impose the condition that each selected random number $y(t)$ is greater than p. Let $F(j + 1)$ be the conditional probability that the random number at the $(j + 1)$th update step is $\geqslant p$, given the previous history of avalanche. Clearly, we have $F(t + 1)$ equals to $[1 - \Phi_t(p)]$. i.e.,

$$F(t+1) = 1 - \prod_{j \in \mathbf{U}(t)} g\left(\frac{p}{x_{\max}(j,t)}\right). \tag{7}$$

Under the condition that $y(t + 1) > p$, the corresponding conditional distribution of $y(t + 1)$ is given by

$$\text{Prob}(y(t+1) \leqslant y | \mathcal{T}_t, y(t+1) \geqslant p) = [\Phi_t(y) - \Phi_t(p)]/F(t+1) \tag{8}$$

with $\phi_t(p)$ defined by equation (4).

The relative weight of a particular history $\mathcal{T}_t$ being realized without the avalanche getting stopped is $\prod_{t'=1}^{t} F(t')$. We calculate the attrition factor $F(t + 1)$ using equation (7). We then randomly select one of the sites in $\mathbf{U}(t)$ for toppling using the correct relative weights described in the appendix. This determines $j(t+1)$. We topple at $j(t+1)$, and at any resulting sites with slope 3, update the values of $x_{\max}(j, t+1)$ at all $j \in \mathbf{U}(t)$. We make the list $\mathbf{U}(t+1)$ of sites with slope $\bar{2}$. This then completes the update step $(t + 1)$. Repeat.

After we start relaxing the unstable configuration $|\bar{2}\bar{2}\bar{2}\ldots\ldots\bar{2}\rangle$, the height at site 1 gradually decreases. At some step of relaxation, the height at first site becomes $h_1 \leqslant h$ for the first time in the course of relaxation. We multiply all the previous factors, $F(t)$'s, up to this step and this product gives

$$W(h) = \prod_{\{t:h_1(t)>h\}} F(t). \tag{9}$$

Averaging $W(h)$ over many initial realizations, we get the probability of height at the first site being less than or equal to h, i.e.,

$$\text{Prob}_L(h_1 \leqslant h) = \langle W(h) \rangle.$$

The estimate of probability of the least mass configuration is obtained by calculating the weight function

$$W_{\min} = \prod_{t=1}^{t_{\max}} F(t) \tag{10}$$

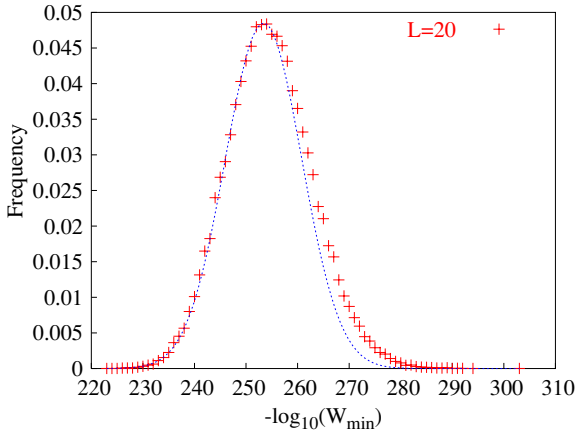

Figure 2. The frequency distribution of $-\log_{10}(W_{\min})$ taking bin size = 1 and starting with 10^6 initial realizations for $L = 20$. We fit the left tail of the data points with a Gaussian distribution with mean $\mu \approx 253.4$ and variance $\sigma^2 \approx 55$.

where $t_{\max}$ is the (random) number of update steps t required to reach the least mass configuration. Since, now one can always find $y(t)$, each avalanche continues until it reaches the minimum mass configuration. For different realizations, we get different values of $W_{\min}$ and, similarly as above, averaging over different values by taking many realizations gives us the probability of the minimum slope.

We illustrate this modified procedure for the simple example given before at the end of section 4 for the unbiassed case. At $t = 0$, we have $\mathbf{U}(0) = \{1, 2, 3, 4, 5, 6\}$, as all sites are unstable. Also, at this stage $x_{\max} = 1$ for all sites. In this case, the probability distribution of $y(1)$ is given by

$$\text{Prob}(y(1) \leqslant y | y(1) > p) = (y^6 - p^6)/(1 - p^6), \qquad \text{for} \quad p \leqslant y \leqslant 1. \tag{11}$$

In generating $y(1)$, as before, we generate a random number z uniformly between 0 and 1 put $y(1) = z^{1/6}$. *If, however,* $y(1) < p$*, we discard this value and choose another* z. We calculate $F(1)$ using equation (7) and get $F(1) = 1 - p^6$. After having generated $y(1)$, we proceed as before to randomly select $j(1)$, topple the sites, and update the $x_{\max}$ array, and determine $\mathbf{U}(1) = \{2, 3, 4, 5, 6\}$. Then, $F(2) = 1 - p^5$, and the probability distribution of $y(2)$ is given by

$$\text{Prob}(y(2) \leqslant y | y(1) > p) = (y^5 - p^5)/(1 - p^5), \qquad \text{for} \quad p \leqslant y \leqslant 1. \tag{12}$$

This is generated by choosing a random number z lying between 0 and 1, and putting $y(2) = z^{1/5}$ if $y(2) > p$, and *rejecting the trial value of* $y(2)$*, if it falls below* p. We then choose one of the sites to topple as $j(2)$ as before. And so on. At each stage, if the trial value of $y(t)$ is less than p, we reject the move and try again.

6. Results

We repeat the above procedure for many realizations and take the average of logarithm of the weight $W(h)$. In figure 2, we have plotted the numerically obtained distribution of $\log W_{\min}$ using 10^6 initial realizations for $L = 20$. It has a peak at $\log W_{\min} \approx -253.4$ and decays rapidly away from the peak. We fit the data point at the left of the peak value to a Gaussian

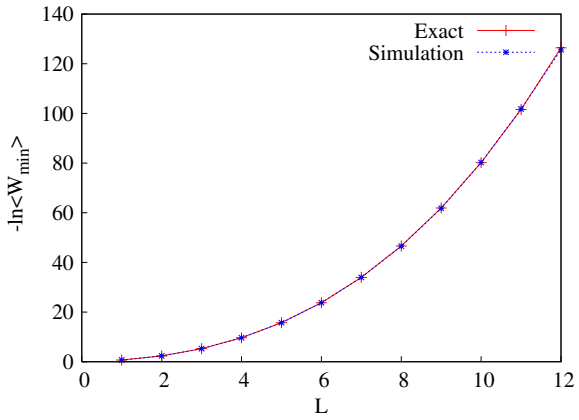

Figure 3. Exact numerical calculation and the Monte Carlo simulation for $L \leqslant 12$: logarithm (base e) of the probability of the minimum slope configuration is plotted against the system size L in the 1D Oslo ricepile model. The data are averaged over 10^6 realizations.

distribution with mean $\mu \approx -253.4$ and variance $\sigma^2 \approx 55$. It should be noted here that our simulation cannot accurately estimate the probabilities of minimal mass configurations for large L and the fractional error may be large, but *the logarithms of the probabilities can be estimated with reasonably small fractional error*. This is similar to the Monte Carlo in equilibrium statistical physics, where thermodynamic quantities like free energy can be estimated well, but not the partition function.

As a check of our simulation algorithm, we calculated the probability of the minimum slope configurations for small L and the numerical values match well with the values obtained from exact numerical calculation using the method in section 3. For example, the probability of the minimum slope, for $L = 5$, is calculated to be 1.475×10^{-7} after averaging the data over 10^6 realizations and the value is correct upto three significant digits. We have compared our results obtained from two procedures, i.e., the Monte Carlo simulation and exact numerical calculation and plotted negative of logarithm of the probabilities against L in figure 3 for $L \leqslant 12$.

For h near L, $W(h)$ is a product of approximately L^3 different factors $F(t)$'s, the logarithm of $W(h)$ is a sum of L^3 random variables. While these variables $F(t)$'s are neither strictly independent nor they are identically distributed, our simulations suggest that correlations between these factors at different times are weak, so that we can expect central-limit-theorem-like result to hold. Then $\log[W(h \simeq L)]$ may be expected to be normally distributed with a mean and variance both proportional to L^3 and $W(h)$ would be log-normally distributed. In fact, the numerically obtained probability distribution function of $\log(W)$ shows significant deviations from Gaussian (figure 2).

Assuming that the distribution of the random variable $X = -\ln[W(h \simeq L)]$ has the form $\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(X-\mu)^2}{2\sigma^2}\right]$, we get the probability of the minimum slope equal to

$$\text{Prob}_L(h_1) = \langle W(h_1)\rangle = \langle \mathrm{e}^{-X}\rangle. \tag{13}$$

Thus, if the Gaussian approximation holds for the distribution,

$$\ln\langle W(h)\rangle \approx \langle \ln W(h)\rangle + \frac{\sigma^2}{2}. \tag{14}$$

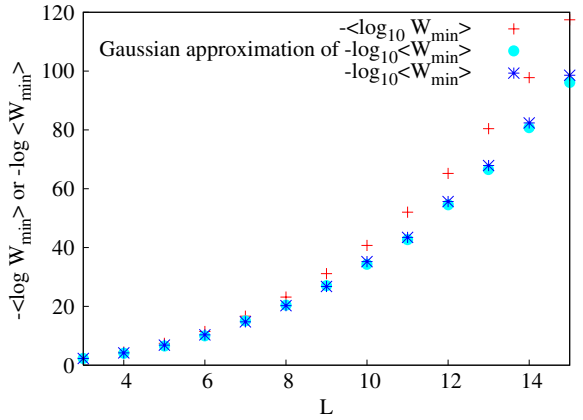

Figure 4. Comparison of $\langle W_{\min}\rangle$, as computed directly from the simulations (blue $\star$) and as estimated by using the Gaussian approximation in equation (14) (light blue bullet) for different system sizes L. Also shown is $-\langle \log_{10} W_{\min}\rangle$ (red +).

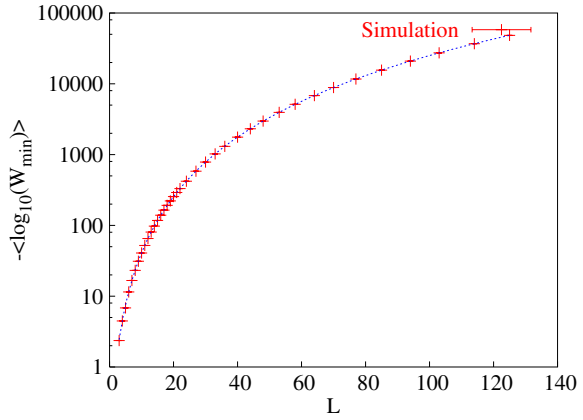

Figure 5. Monte Carlo estimate of $\langle \log W_{\min}\rangle$ plotted against system size L in the 1D Oslo ricepile model. The data points are fitted with $ax^3 + bx^2 + cx$ where $a = 0.0234$, $b = 0.16$ and $c = 0.1$.

This estimate need not be precise as the terms contribute significantly to $\langle W(h \simeq L)\rangle$ are in the tail of the distributions and central limit theorem need not hold there.

However, our numerical estimate indicates that this approximation is indeed very good. This is because the deviations from the Gaussian behaviour are stronger for smaller values of W, but these do not contribute much to $\langle W\rangle$. For example, from the simulation for $L = 15$, we get $\mu \approx -270$, $\sigma^2 \approx 95$, $\ln\langle W_{\min}\rangle \approx -229$. The Gaussian approximation to distribution of $\ln W_{\min}$ would give $\ln\langle W_{\min}\rangle \approx -223$. In figure 4, we have compared the actual values of $\log_{10}\langle W_{\min}\rangle$ from the simulation with the estimate from the Gaussian approximation for different values of the system size L.

In figure 5, we have plotted negative of $\langle \log W_{\min}\rangle$ as a function of L and fitted it with a curve $ax^3 + bx^2 + cx$. We get a god fit for $a = 0.0234 \pm 0.0001$, $b = 0.16 \pm 0.05$ and $c = 0.1 \pm 0.2$.

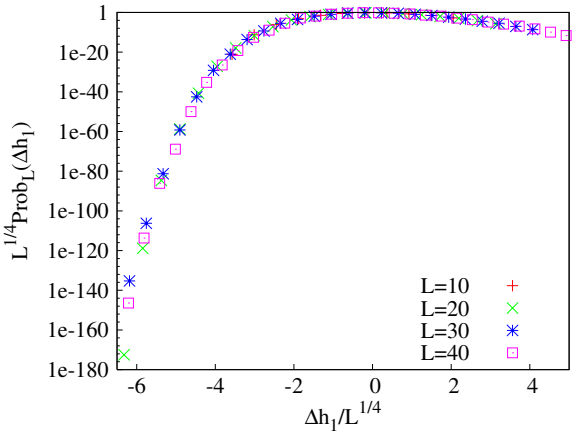

Figure 6. Scaling collapse for the probability distribution of height at site 1 for systems of size $L = 10, 20, 30, 40$. $L^{1/4}\mathrm{Prob}_L(\Delta h_1)$ has been plotted against $(h_1 - \bar{h}_1)/L^{1/4}$. Each point is averaged over 10^6 realizations.

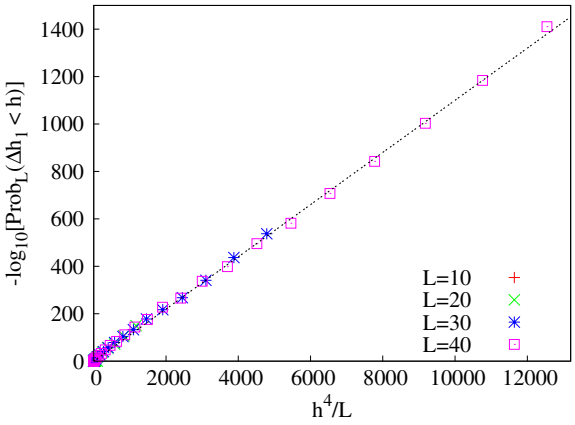

Figure 7. $\log_{10}[\mathrm{Prob}_L(\Delta h_1/L^{1/4} < -x)]$, for $x > 0$, has been plotted against the scaling variable x^4 where $\Delta h_1 = (h_1 - \bar{h}_1)$. The data points are fitted with a straight line $0.11x$ and it shows that the scaling function $g(-x)$ varies as $\exp(-\kappa x^4)$ for $x \gg 1$.

Now we calculate the full probability distribution $\mathrm{Prob}_L(h_1)$ of height h_1 at site 1. We take the average of this product over many realizations and also study the distribution $\mathrm{Prob}_L(h_1 = h)$ as a function of h, for $L = 10, 20, 30, 40$. The data are averaged over 10^6 initial realizations. In figure 6, we get a good scaling collapse by plotting $L^{\omega}\mathrm{Prob}_L(h_1 = h)$ against the scaling variable $(h_1 - \bar{h}_1)/L^{\omega}$ where $\omega \approx 1/4$. Therefore, $\mathrm{Prob}_L(h_1 = h)$ has a scaling form $\mathrm{Prob}_L(h_1) = L^{-1/4}g[(h_1 - \langle h_1 \rangle)/L^{1/4}]$ at the central region as well as at the tail. We see from the scaling plot that the scaling function is highly asymmetric about the origin.

Since the probability of minimum slope configurations varies as $\exp(-\kappa' L^3)$ for large L where $\kappa' > 0$ and the height fluctuation $\Delta h_1 = (h_1 - \bar{h}_1)$ scales with system sizes as $L^{1/4}$, the scaling function $g(x)$ must vary as $\exp(-\kappa x^4)$, where $\kappa > 0$ is some other constant. In figure 7, we plot logarithm of $\mathrm{Prob}_L(\Delta h_1/L^{1/4} < -x)$ versus x^4 for $x > 0$ (i.e., fluctuations below average height). The data give a reasonably good fit to a straight line with slope 0.11.

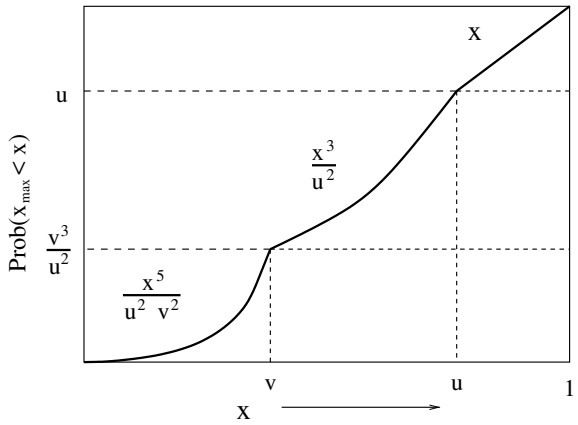

Figure 8. The probability Prob($x_{\max} < x$) of $x_{\max}$ being less than x versus x.

7. Summary

We have performed a Monte Carlo simulation using importance sampling to study large deviations in the one-dimensional Oslo ricepile model. We estimated probabilities of large height fluctuations of the pile and these probabilities are of order 10^{-1000} or even smaller (see figure 7). We have shown that logarithm of the probability of large negative height fluctuation $\Delta h = -\alpha L$ varies as $-\alpha^4 L^3$ for large L. We also calculated numerically the full probability distribution of the height of the pile and found that it has scaling form $L^{-1/4} g(\Delta h / L^{1/4})$, with $\log[g(x)]$ varying as $-x^4$ for large negative x.

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Appendix

We illustrate the procedure for generating the largest among $x(j,t)$'s, where $j \in \mathbf{u}$, using the algorithm given here. Consider independent random variables x_1, x_2, x_3, x_4, x_5 which are known to be uniformly distributed between the respective intervals $x_1 \in [0,1], x_2 \in [0,u], x_3 \in [0,u], x_4 \in [0,v]$ and $x_5 \in [0,v]$. We may take $1 \geqslant u \geqslant v$ without loss of generality. Then, the cumulative probability distribution of y, the largest among these random numbers, is given by

$$\begin{aligned} \text{Prob}(y \leqslant x) &= x, && \text{for} \quad u \leqslant x \leqslant 1, \\ &= x^3/u^2, && \text{for} \quad v \leqslant x \leqslant u, \\ &= x^5/(u^2 v^2), && \text{for} \quad 0 \leqslant x \leqslant v. \end{aligned} \tag{A.1}$$

The distribution is drawn schematically in figure 8. To generate a variable y with this distribution, we use the following algorithm: generate a number z randomly between 0 and 1. Then, following cases are possible:

(1) If $u < z \leqslant 1$, we choose the largest random number to be $y = z$ and the maximum is surely x_5.

(2) If $v^3/u^2 < z \leqslant u$, we choose the largest random number to be $y = (zu^2)^{1/3}$ and the maximum is chosen from x_3, x_4 and x_5 with probability $1/3$ each.
(3) If $0 \leqslant z \leqslant v^3/u^2$, we choose the largest random number to be $y = (zu^2v^2)^{1/5}$ and the maximum is chosen from x_1, x_2, x_3, x_4 and x_5 with probability $1/5$ each.

In the modified procedure, the only change made is that we reject the value of y, if it is less than p, and choose a new random number z, and proceed as before. Probability distributions for the maximum of more variables can be obtained similarly.

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